TWO CLASSES OF SOLUTIONS OF THE FLUID AND GAS MECHANICS EQUATIONS AND THEIR CONNECTION TO TRAVELING WAVE THEORY

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A description is given of two classes of spatial fluid and gas motions possessing large functional arbitrariness and characterized by the property of linearity of the main flow parameters over part of the spatial coordinates. The constructed classes of solutions permit taking account of such properties of a continuous medium as heat and electrical conductivities for a gas, and viscosity and electrical conductivity for a fluid in the Boussinesq approximation. The relation of the described flows to the theory of traveling waves of the rank of three-triple waves is investigated for an inviscid gas. Definite systems of equations describing new types of triple vortex waves possessing functional arbitrariness are obtained as specifications of the original classes of flows. Series of exact solutions are constructed.

1. Classes of solutions of nonstationary spatial equations of motion of an incompressible fluid and of gas dynamics when the velocity vector components are linear functions of all the spatial coordinates are well known and were studied in [1, 2] for an incompressible medium and in [3, 4] for a gas. In group terminology such flow classes are H-invariant solutions [5], they have found a number of meaningful interpretations [4]. The question of the existence of spatial fluid and gas flows with a linear dependence of the velocity vector components $u_k(x_1, x_2, x_3, t)$ on part of the spatial coordinates (one or two) is not trivial.

The relationships

$$u_k = l_k(x_1, t)x_2 + f_k(x_1, t)x_3 + g_k(x_1, t) \ (k = 1, 2, 3) \tag{1.1}$$

should be satisfied for flows of class I in flows of this kind and

$$u_k = f_k(x_1, x_2, t)x_3 + g_k(x_1, x_2, t) \ (k = 1, 2, 3) \tag{1.2}$$

for flows of class II, where ℓ_k, f_k, and g_k are functions to be determined.

The equations of motion of a medium are usually successfully reduced to a form when each component contains a polynomial nonlinearity of the unknown functions not higher than the quadratic. Appropriate thermodynamic functions are also represented by expressions of the type (1.1) and (1.2) (sometimes they can contain components quadratic in x_2 and x_3 also).

After substitution of representations of the type (1.1) and (1.2) into such a system of equations polynomial expressions in the variables x_2, x_3, or x_3 with coefficients containing differential aggregates of ℓ_k, f_k, and g_k are obtained for all the unknown functions. Equating all these coefficients to zero, we obtain a strongly overdefined system of m partial differential equations for n unknown functions of ℓ_k, f_k, and g_k and analogous coefficients entering in the thermodynamic functions ($m > n$).

A general analysis of the compatibility of such systems is quite awkward and complex. Its execution and the determination of the arbitrariness in the solutions have not been successful except in several particular cases. However, it turns out to be possible to indicate simple sufficient compatibility conditions for the systems obtained [6-8] for a number of cases with the above-mentioned properties of the medium taken into account. The definite systems of ℓ equations constructed here (ℓ is the number of equations, examples will be presented below for an inviscid gas) possess broad functional arbitrariness. Although these systems have still been investigated perfectly inadequately, they have already found a number of applications in the solution of specific gas dynamic problems [9-11], particularly in the investigation of the dynamics of rotating vortical gas flows, and also permitted the construction of classes of exact solutions.

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TABLE 1

Motion class		Inviscid gas			Viscous fluid	
		ideal	heat conducting	electrical conducting	in the Boussinesq approxima.	taking elec. conductivity into account
I (1.1)	m	24	46	67	27	48
	n	12	15	24	18	27
	l	8	9	11	11	13; 5
II (1.2)	m	12	19	30	14	25
	n	8	10	16	11	17
	l	5	6	7	7	8; 8

Numerical values of m, n, and ℓ for several fundamental types of media are presented in Table 1. Let us note that the number ℓ agrees with the number of arbitrary functions of one or two arguments on which the appropriate class of motions depends if all the equations of the system obtained are of first order. Two different modifications of definite systems occur for an electrically conductive viscous fluid in the Boussinesq approximation.

The following motion properties are common for all the cases considered: vorticity, nonisoenergeticity, degeneracy in the general case of the velocity hodograph. The group nature of such solutions is as yet unclear. The structure of the obtained systems of definite equations describing the motion classes I and II is similar to the structure of the original fluid or gas motion equations when the dimensionality of the space of independent variables diminishes by one, but mass forces dependent nonlinearly on the unknown functions enter into the right sides of the system obtained. Let us note that, in the most general case of viscous compressible gas flows, sufficient conditions for compatibility resulting in nontrivial definite systems describing meaningful motion classes have still not been obtained successfully.

2. Let us examine the question of the relation of the motion classes I and II to the traveling wave theory. At this time traveling wave theory has obtained broad development for the equations of gas dynamics. The most complete survey of available results is contained in [12]. Traveling waves are here understood to be a class of solutions of the equations of gas dynamics characterized by the fact that a domain of lower dimensionality r (r = 1, 2, 3) in the hodograph space for the velocities u_1, u_2, u_3 corresponds to the four-dimensional domain of the original physical space x_1, x_2, x_3, t for them. The quantity r is called the rank of the wave.

In practice, the case r = 1, the so-called simple waves, has been studied completely. For r = 2 (double waves) gas dynamic flows have been studied most fully under the assumption of stream potentiality, as have also been double waves with rectilinear generatrices that are characterized by the fact that the fundamental gas dynamic parameters retain a constant value along a certain set of lines in the original physical space. Governing equations are obtained for these types of double waves and a functional arbitrariness existing in the solutions is established. Recently, a sufficiently complete classification of double waves of a general type has been given for two-dimensional plane-parallel nonstationary flows [13]. As regards the triple waves (r = 3), their regular description (by using definite systems of differential equations) is absent even for potential flows. Only individual classes of exact solutions of the potential triple wave type [12] (although sufficiently broad possessing functional arbitrariness) and one particular class of triple vortical waves [6] have been constructed.

The difficulties in studying waves of ranks two and three that are partially invariant solutions from the group viewpoint [14] are associated with the necessity to investigate complex and awkward overdefined systems of partial differential equations. Despite the available general approaches to the solution of such problems (the Cartan algorithm and its modifications), their specific realization is associated with large analytic calculations and even by using specialized programs to perform analytic calculations on electronic computers did not result in success, particularly in the investigation of the compatibility of systems of potential triple wave equations. In fact, each serious advance in the theory of multiple traveling waves required a specialized analytical study in appropriate spaces of dependent and independent variables.

The purpose of the following sections is to describe new classes of vortical triple waves that occur as separate specializations of more degenerate gas dynamic flows of classes I and II (with a nondegenerate velocity hodograph) examined in Sec. 1. Narrowing these motion classes when a condition of degeneracy of the velocity hodograph is imposed in addition, results as before in new overdefined systems of equations. Nevertheless, although it is quite difficult to perform a general analysis of compatibility, sufficient conditions can be indicated when the obtained overdefined systems reduce to defined systems and, therefore, new descriptions can be found for vortical traveling waves of rank three with broad functional arbitrariness.

3. Let us consider traveling waves in the class of flows with the linearity property in two space coordinates. We write the system of gas dynamics equations for the functions **u** (the velocity vector), $Q = \rho^{\gamma-1}$ (ρ is the density), W (the entropy function, the equation of state has the form $p = W\rho^{\gamma}$, p is the pressure, γ the adiabatic index) in the form

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\nabla)\,\mathbf{u} + Q\,\mathrm{grad}\,W + \frac{\gamma}{\gamma-1}W\,\mathrm{grad}\,Q = 0; \tag{3.1}$$

$$\frac{\partial Q}{\partial t} + (\mathbf{u}\,\mathrm{grad}\,Q) + (\gamma-1)\,Q\,\mathrm{div}\,\mathbf{u} = 0; \tag{3.2}$$

$$\frac{\partial W}{\partial t} + (\mathbf{u}\,\mathrm{grad}\,W) = 0. \tag{3.3}$$

Solutions of the system (1.1)-(1.3) are constructed in [6] in the form

$$\begin{aligned} u_k &= l_k(x_1, t)x_2 + f_k(x_1, t)x_3 + g_k(x_1, t)\ (k = 1, 2, 3), \\ Q &= l(x_1, t)x_2 + f(x_1, t)x_3 + g(x_1, t), \\ W &= L(x_1, t)x_2 + F(x_1, t)x_3 + G(x_1, t). \end{aligned} \tag{3.4}$$

If we set

$$l_1 = f_1 = l = f = L = F = 0, \tag{3.5}$$

in (3.4), then the remaining nine unknown functions ℓ_2, ℓ_3, f_2, f_3, g_1, g_2, g_3, g, and G satisfy the following system of nine equations:

$$\frac{\partial l_i}{\partial t} + g_1\frac{\partial l_i}{\partial x_1} + l_2 l_i + l_3 f_i = 0, \quad i = 2, 3; \tag{3.6}$$

$$\frac{\partial f_i}{\partial t} + g_1\frac{\partial f_i}{\partial x_1} + f_2 l_i + f_3 f_i = 0, \quad i = 2, 3; \tag{3.7}$$

$$\frac{\partial g_i}{\partial t} + g_1\frac{\partial g_i}{\partial x_1} + g_2 l_i + g_3 f_i = 0, \quad i = 2, 3; \tag{3.8}$$

$$\frac{\partial g_1}{\partial t} + g_1\frac{\partial g_1}{\partial x_1} + g\frac{\partial G}{\partial x_1} + \frac{\gamma}{\gamma-1}G\frac{\partial g}{\partial x_1} = 0; \tag{3.9}$$

$$\frac{\partial g}{\partial t} + g_1\frac{\partial g}{\partial x_1} + (\gamma-1)\,g\left(\frac{\partial g_1}{\partial x_1} + l_2 + f_3\right) = 0; \tag{3.10}$$

$$\frac{\partial G}{\partial t} + g_1\frac{\partial G}{\partial x_1} = 0. \tag{3.11}$$

As is known, that solution of the system of equations (3.1)-(3.3) for which the rank of the Jacobi matrix A for u_k, Q, and W equals r is called a traveling wave of rank r. In this case the matrix A has the form

$$A = \begin{pmatrix} \dfrac{\partial g_1}{\partial x_1} & 0 & 0 & \dfrac{\partial g_1}{\partial t} \\ \dfrac{\partial l_2}{\partial x_1}x_2 + \dfrac{\partial f_2}{\partial x_1}x_3 + \dfrac{\partial g_2}{\partial x_1} & l_2 & f_2 & \dfrac{\partial l_2}{\partial t}x_2 + \dfrac{\partial f_2}{\partial t}x_3 + \dfrac{\partial g_2}{\partial t} \\ \dfrac{\partial l_3}{\partial x_1}x_2 + \dfrac{\partial f_3}{\partial x_1}x_3 + \dfrac{\partial g_3}{\partial x_1} & l_3 & f_3 & \dfrac{\partial l_3}{\partial t}x_2 + \dfrac{\partial f_3}{\partial t}x_3 + \dfrac{\partial g_3}{\partial t} \\ \dfrac{\partial g}{\partial x_1} & 0 & 0 & \dfrac{\partial g}{\partial t} \\ \dfrac{\partial G}{\partial x_1} & 0 & 0 & \dfrac{\partial G}{\partial t} \end{pmatrix}.$$

It is clear that if $\ell_2 f_3 - \ell_3 f_2 \neq 0$ and the functions g_1 and g or g and G are functionally independent, then r = 4 and (3.4) generally define flows of general type.

Let us first examine the case when the hodograph for the flows (3.4) is degenerate and

$$z = l_2/l_3 = f_2/f_3. \tag{3.12}$$

Then if $(\ell_2, \ell_3) \neq (0, 0)$, and g and g_1 are functionally independent, r = 3 and the triple wave case is realized.

Combining linear combinations from the pairs of relationships (3.6) and (3.7), with the coefficients ℓ_3 and $-\ell_2$, f_3 and $-f_2$, respectively, we have

$$\begin{aligned} l_3\frac{\partial l_2}{\partial t} - l_2\frac{\partial l_3}{\partial t} + g_1\left(l_3\frac{\partial l_2}{\partial x_1} - l_2\frac{\partial l_3}{\partial x_1}\right) = 0, \\ f_3\frac{\partial f_2}{\partial t} - f_2\frac{\partial f_3}{\partial t} + g_1\left(f_3\frac{\partial f_2}{\partial x_1} - f_2\frac{\partial f_3}{\partial x_1}\right) = 0. \end{aligned} \tag{3.13}$$

Both equations of (3.13) reduce by using (3.12) to one

$$\partial z/\partial t + g_1 \partial z/\partial x_1 = 0. \tag{3.14}$$

Therefore, just two equations can be kept in place of (3.6) and (3.7)

$$\partial l_2/\partial t + g_1 \partial l_2/\partial x_1 + l_2^2 + l_3 f_2 = 0; \tag{3.15}$$

$$\partial f_2/\partial t + g_1 \partial f_2/\partial x_1 + f_2(l_2 + f_3) = 0. \tag{3.16}$$

By setting $\ell_3 = \ell_2 f_3 f_2^{-1}$ (3.12) in this case, we obtain a definite system of eight equations (3.15), (3.16), (3.14), (3.8)-(3.11) for the functions ℓ_2, f_2, f_3, g_1, g_2, g_3, g, and G that describes the class of vortical nonstationary nonisentropic spatial triple waves that possess arbitrariness in eight functions of one independent argument. The subsystem of equations (3.14)-(3.16), (3.9)-(3.11) is here independent. After its solution the functions g_2 and g_3 are found from the system of two linear equations (3.8).

Now, let $\ell_2 f_3 - \ell_3 f_2 \neq 0$. Since the fourth-order determinants should vanish, we obtain the functional dependences

$$g = F(g_1),\ G = G(g). \tag{3.17}$$

By virtue of (3.17) the equation of state should correspond to a barotropic gas and, later, for simplicity, we set $G = G_0$ = const, i.e., consider an isentropic case. Then (3.9) acquires the form

$$\frac{\partial g_1}{\partial t} + \omega(g_1)\frac{\partial g_1}{\partial x_1} = 0, \quad \omega(g_1) = g_1 + \frac{\gamma}{\gamma - 1} G_0 F', \tag{3.18}$$

and its general integral can be represented in the form

$$x_1 = \omega(g_1)t + \Gamma(g_1), \tag{3.19}$$

where Γ is an arbitrary function. As already noted, after the functions ℓ_2, ℓ_3, f_2, and f_3 have been determined, the functions g_2 and g_3 are also found for this case by integrating the linear system (3.8).

Therefore, the situation reduces to analysis of the compatibility of a system of five equations for four functions ℓ_2, ℓ_3, f_2, f_3 in whose coefficients are two arbitrary functions F and Γ of one argument. Here (3.10) takes the form

$$l_2 + f_3 = -\left(1 - \frac{\gamma}{(\gamma-1)^2} G_0 \frac{F'^2}{F}\right)(\omega' t + \Gamma')^{-1}, \tag{3.20}$$

i.e., the function f_3 can be expressed in terms of ℓ_2 and the system (3.6), (3.7) is subject to further analysis. Its complete analysis has not yet been successfully performed. Let us examine a particular case and let us show that the set of solutions of this overdefined system is not empty and possesses arbitrariness in at least several arbitrary constants. Taking (3.8) into account the class of triple waves with arbitrariness in two functions of a single argument will thereby be constructed.

Let us assume that the following dependences hold:

$$\omega = a = \text{const},\ \xi = x_1 - at,\ l_k = L_k(\xi),$$
$$f_k = F_k(\xi),\ g_1 = G_1(\xi) + a,\ g = H(\xi). \tag{3.21}$$

Then (3.6), (3.7), (3.9), and (3.10) reduce to the form

$$G_1L_2' + L_2^2 + L_3F_2 = 0,\quad G_1L_3' + L_3(L_2 + F_3) = 0,$$
$$G_1F_2' + F_2(L_2 + F_3) = 0,\quad G_1F_3' + F_2L_3 + F_3^2 = 0, \tag{3.22}$$
$$\frac{1}{2}G_1^2 + \frac{\gamma}{\gamma - 1}G_0H = C_1 = \text{const},\quad G_1H' + (\gamma - 1)H(G_1' + L_2 + F_3) = 0.$$

But such a system of ordinary equations (3.22) was already considered in [9] when studying stationary flows (ξ plays the part of the stationary coordinate x_3') and was integrated in quadratures. Its solution depends on six arbitrary constants.

The system of equations (3.8) for the functions g_2 and g_3 has the form

$$\frac{\partial g_2}{\partial t} + (G_1 + a)\frac{\partial g_2}{\partial x_1} + g_2L_2 + g_3F_2 = 0,$$
$$\frac{\partial g_3}{\partial t} + (G_1 + a)\frac{\partial g_3}{\partial x_1} + g_2L_3 + g_3F_3 = 0. \tag{3.23}$$

Let us go over to new independent variables t', ξ by setting

$$\xi = x_1 - at,\quad t' = \int\frac{d\xi}{G_1(\xi)} - t. \tag{3.24}$$

Then (3.23) reduces to the form

$$G_1(\xi)\frac{\partial g_2}{\partial \xi} + g_2L_2(\xi) + g_3F_2(\xi) = 0,$$
$$G_1(\xi)\frac{\partial g_3}{\partial \xi} + g_2L_3(\xi) + g_3F_3(\xi) = 0. \tag{3.25}$$

Eliminating g_3 from (3.25), we obtain the following actually ordinary linear differential equation with variable ξ for g_2 by using (3.22):

$$G_1^2\frac{\partial^2 g_2}{\partial \xi^2} + G_1(G_1' + 2L_2 + 2F_3)\frac{\partial g_2}{\partial \xi} + 2g_2(L_2F_3 - L_3F_2) = 0. \tag{3.26}$$

The general solution of (3.26) can be represented as $g_2 = A_1(t')\Phi_1(\xi) + A_2(t')\Phi_2(\xi)$, where Φ_1 and ϕ_2 are fundamental solutions of (3.26) while A_1 and A_2 are arbitrary functions of t'.

Therefore, a class of solutions of the type of nonstationary isentropic vortical spatial triple waves has been constructed with arbitrariness in two functions of a single argument. The functions g_2 and g_3 are analogous to known "arrangement" functions in the theory of flows with a degenerate hodograph [12].

4. Let us consider the linear solutions in one space coordinate.

Solutions of the system (3.1)-(3.3) of the following form are constructed in [6]:

$$u_k = f_k(x_1, x_2, t)x_3 + g_k(x_1, x_2, t)\ (k = 1, 2, 3),$$
$$Q = f(x_1, x_2, t)x_3 + g(x_1, x_2, t), \tag{4.1}$$
$$W = F(x_1, x_2, t)x_3 + G(x_1, x_2, t).$$

If

$$f_1 = f_2 = f = F = 0, \tag{4.2}$$

in (4.1), then the remaining six unknown functions g_1, g_2, g_3, f_3, g, and G will satisfy a system of six equations [$v = (f_1, f_2)$]:

$$
\begin{aligned}
&\frac{\partial f_3}{\partial t}+(\mathbf{v}\operatorname{grad} f_3)+f_3^2=0,\\
&\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\nabla)\,\mathbf{v}+g\operatorname{grad}G+\frac{\gamma}{\gamma-1}G\operatorname{grad}g=0,\\
&\frac{\partial g_3}{\partial t}+(\mathbf{v}\operatorname{grad}g_3)+f_3g_3=0,\\
&\frac{\partial g}{\partial t}+(\mathbf{v}\operatorname{grad}g)+(\gamma-1)\,g\,(f_3+\operatorname{div}\mathbf{v})=0,\\
&\frac{\partial G}{\partial t}+(\mathbf{v}\operatorname{grad}G)=0.
\end{aligned}
\tag{4.3}
$$

In this case the matrix is

$$
A=\begin{pmatrix}
\frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & 0 & \frac{\partial g_1}{\partial t}\\
\frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} & 0 & \frac{\partial g_2}{\partial t}\\
x_3\frac{\partial f_3}{\partial x_1}+\frac{\partial g_3}{\partial x_1} & x_3\frac{\partial f_3}{\partial x_2}+\frac{\partial g_3}{\partial x_2} & f_3 & x_3\frac{\partial f_3}{\partial t}+\frac{\partial g_3}{\partial t}\\
\frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & 0 & \frac{\partial g}{\partial t}\\
\frac{\partial G}{\partial x_1} & \frac{\partial G}{\partial x_2} & 0 & \frac{\partial G}{\partial t}
\end{pmatrix}.
$$

Let $r < 4$ and $f_3 \neq 0$ (if $f_3 = 0$ then a slightly interesting case is obtained where all the gas dynamic quantities are independent of x_3). Then we find the following functional dependences at once from the form of A:

$$
g=M(g_1,\ g_2),\ G=N(g_1,\ g_2).
\tag{4.4}
$$

Analysis of the compatibility of the overdefined system of equations obtained from (4.3) and (4.4) is quite complex although a number of intermediate integrals can be obtained sufficiently easily. Despite the fact that the class of solutions of the form (4.1) is more general than the solutions (3.4), nevertheless, solutions of the form (3.4) are already not included because of the additional constraints (4.2) that narrow this class. The question of the meaningfulness of the imbedding of flows of the traveling-wave type in the class (4.1) and (4.2) remains open in the general case. For instance, let us present an example showing that the class of triple waves of the type (4.1) and (4.2) is not empty and possesses arbitrariness to a lesser degree in five functions of a single argument and one function of two independent variables.

Let us consider that the following functional relationships are valid in the system (4.3):

$$
\begin{aligned}
&g_k=G_k(\xi_1,\ \xi_2)+a_k,\ \xi_k=x_k-a_kt\ (k=1,\ 2),\\
&g=M(\xi_1,\ \xi_2),\ G=N(\xi_1,\ \xi_2),\ f_3=F(\xi_1,\ \xi_2).
\end{aligned}
\tag{4.5}
$$

Then five equations of system (4.3) (except the equation for g_3) for the five functions G_1, G_2, M, N, and F can be written in the form

$$
\begin{aligned}
&G_1\frac{\partial G_k}{\partial \xi_1}+G_2\frac{\partial G_k}{\partial \xi_2}+M\frac{\partial N}{\partial \xi_k}+\frac{\gamma}{\gamma-1}N\frac{\partial M}{\partial \xi_k}=0\quad (k=1,2),\\
&G_1\frac{\partial F}{\partial \xi_1}+G_2\frac{\partial F}{\partial \xi_2}+F^2=0,\\
&G_1\frac{\partial M}{\partial \xi_1}+G_2\frac{\partial M}{\partial \xi_2}+(\gamma-1)\,M\left(F+\frac{\partial G_1}{\partial \xi_1}+\frac{\partial G_2}{\partial \xi_2}\right)=0,\quad G_1\frac{\partial N}{\partial \xi_1}+G_2\frac{\partial N}{\partial \xi_2}=0.
\end{aligned}
$$

After having solved this system that contains arbitrariness in five functions of one variable, a linear first-order equation remains for the determination of g_3:

$$
\frac{\partial g_3}{\partial t}+(G_1+a_1)\frac{\partial g_3}{\partial x_1}+(G_2+a_2)\frac{\partial g_3}{\partial x_2}+Fg_3=0.
$$

Therefore, a class of triple waves with the above-mentioned arbitrariness has been constructed (if G_1 and G_2 are functionally independent).

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